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International Journal of Heat and Mass Transfer 46 (2003) 2695–2700
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Unsteady flow of a non-Newtonian fluid above a rotating disk with heat transfer

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Received 29 November 2001; received in revised form 4 October 2002

Abstract

The unsteady flow of an incompressible viscous non-Newtonian fluid above an infinite rotating disk is studied with heat transfer. The effect of the non-Newtonian fluid characteristics on the velocity and temperature distributions as well as the heat transfer is considered. Numerical solutions for the non-linear partial differential equations which govern the hydrodynamics and energy transfer are obtained.

1. Introduction

In 1921 von Karman [1] has studied the fluid flow due to an infinite rotating disk. Later, Cochran [2] obtained asymptotic solutions for the steady hydrodynamic problem formulated by von Karman. Benton [3] extended the problem to flow started impulsively from rest. The steady flow of a non-Newtonian fluid due to a rotating disk was considered by Mithal [4]. The solutions obtained were valid for small values of the parameter which describes the non-Newtonian behavior. Srivastava [5] extended the problem to the case where the flow is between two infinite disks, one is rotating and the other is at rest.

The problem of heat transfer from a rotating disk maintained at a constant temperature was first considered by Millsaps and Pohlhausen [6] for a variety of Prandtl numbers in the steady state. Sparrow and Gregg [7] studied the steady state heat transfer from a rotating disk maintained at a constant temperature to fluids at any Prandtl number. Later, many authors have studied the heat transfer near a rotating disk considering different thermal conditions [8–12].

In the present work, the unsteady laminar flow of a viscous incompressible non-Newtonian fluid due to the uniform rotation of an infinite disk is studied with heat transfer. The temperature of the disk is impulsively changed and then maintained at a constant value. Due to the difference in temperature between the ambient and the surface of the disk heat transfer takes place. The governing non-linear partial differential equations are integrated numerically using the finite difference approximations. The effect of the characteristics of the non-Newtonian fluid on the unsteady flow and heat transfer is discussed.

2. Basic equations

Let the disk lie in the plane $z = 0$ and the space $z > 0$ is equipped by a viscous incompressible non-Newtonian fluid. The motion is started impulsively from rest due to the rotation of the disk with a constant angular velocity ω about the axis of the disk. The equations of unsteady motion are given by

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = \frac{\partial \tau_r^r}{\partial r} + \frac{\partial \tau_r^z}{\partial z} + \frac{\tau_r^r - \tau_\phi^\phi}{r}, \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right) = \frac{\partial \tau_\phi^r}{\partial r} + \frac{\partial \tau_\phi^z}{\partial z} + \frac{2\tau_\phi^r}{r}, \tag{3}$$

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doi:10.1016/S0017-9310(03)00029-2

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=\frac{\partial \tau_z^r}{\partial r}+\frac{\partial \tau_z^z}{\partial z}+\frac{\tau_z^r}{r}, \quad (4)$$

where u, v, and w are velocity components in the directions of increasing r, ϕ, and z respectively, and ρ is the density of the fluid. The non-Newtonian fluid considered in the present paper is that for which the stress tensor τ_j^i is related to the rate of strain tensor e_j^i as [4]

$$\tau_j^i = 2\mu e_j^i + 2\mu_c e_k^i e_j^k - p\delta_j^i, \quad e_j^j = 0, \quad (5)$$

where p is denoting the pressure, μ is the coefficient of viscosity and μ_c is the coefficient of cross viscosity.

By introducing von Karman transformations [1],

$$u = r\omega F, \quad v = r\omega G, \quad w = (\omega v)^{1/2}H,$$
$$z = (v/\omega)^{1/2}\zeta, \quad p - p_\infty = -\rho v\omega P,$$

where ζ is a non-dimensional distance measured along the axis of rotation, F, G, H, and P are non-dimensional functions of ζ and t, and v is the kinematic viscosity of the fluid, $v = \mu/\rho$. With these definitions, Eqs. (1)–(5) take the form

$$\frac{\partial H}{\partial \zeta} + 2F = 0, \quad (6)$$

$$\frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial \zeta^2} + H\frac{\partial F}{\partial \zeta} + F^2 - G^2 + \frac{1}{2}K\left(\left(\frac{\partial F}{\partial \zeta}\right)^2 + 3\left(\frac{\partial G}{\partial \zeta}\right)^2 + 2F\frac{\partial^2 F}{\partial \zeta^2}\right) = 0, \quad (7)$$

$$\frac{\partial G}{\partial t} - \frac{\partial^2 G}{\partial \zeta^2} + H\frac{\partial G}{\partial \zeta} + 2FG - K\left(\frac{\partial F}{\partial \zeta}\frac{\partial G}{\partial \zeta} - F\frac{\partial^2 G}{\partial \zeta^2}\right) = 0, \quad (8)$$

$$\frac{\partial H}{\partial t} - \frac{\partial^2 H}{\partial \zeta^2} + H\frac{\partial H}{\partial \zeta} + \frac{7}{2}K\frac{\partial H}{\partial \zeta}\frac{\partial^2 H}{\partial \zeta^2} - \frac{dP}{d\zeta} = 0, \quad (9)$$

where K is the parameter that describes the non-Newtonian behavior, $K = \mu_c\omega/\mu$. The initial and boundary conditions for the velocity problem are given by

$$F(0,\zeta) = 0, \quad G(0,\zeta) = 0, \quad H(0,\zeta) = 0, \quad (10a)$$

$$F(t,0) = 0, \quad G(t,0) = 1, \quad H(t,0) = 0, \quad (10b)$$

$$F(t,\infty) = 0, \quad G(t,\infty) = 0, \quad P(t,\infty) = 0. \quad (10c)$$

The initial conditions are given by Eq. (10a). Eq. (10b) indicates the no-slip condition of viscous flow applied at the surface of the disk. Far from the surface of the disk, all fluid velocities must vanish aside the induced axial component as indicated in Eq. (10c). The above system of Eqs. (6)–(9) with the prescribed initial and boundary conditions given by Eq. (10) are sufficient to solve for the three components of the flow velocity and the pressure distribution.

Due to the difference in temperature between the wall and the ambient fluid heat transfer takes place. The energy equation, by neglecting the dissipation terms, takes the form [6,7],

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right) - k\frac{\partial^2 T}{\partial z^2} = 0. \quad (11)$$

The initial and boundary conditions for the energy problem are that the temperature is changed impulsively from rest and, by continuity considerations, it equals T_w at the surface of the disk. At large distances from the disk, T tends to T_∞ where T_∞ is the temperature of the ambient fluid.

In terms of the non-dimensional variable $\theta = (T - T_\infty)/(T_w - T_\infty)$ and using von Karman transformations equation (11) takes the form,

$$\frac{\partial \theta}{\partial t} - \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \zeta^2} + H\frac{\partial \theta}{\partial \zeta} = 0, \quad (12)$$

where Pr is the Prandtl number given by, $Pr = c_p\mu/k$. The initial and boundary conditions in terms of θ are expressed as

$$\theta(0,\zeta) = 0, \quad \theta(t,0) = 1, \quad \theta(t,\infty) = 0. \quad (13)$$

The heat transfer from the disk surface to the fluid is computed by application of Fourier's law; $q = -k(\partial T/\partial z)_w$. Introducing the transformed variables, the expression for q becomes

$$q = -k(T_w - T_\infty)\left(\frac{\omega}{v}\right)^{1/2}\frac{\partial \theta(t,0)}{\partial \zeta}.$$

By rephrasing the heat transfer results in terms of a Nusselt number defined as, $Nu = q(v/\omega)^{1/2}/k(T_w - T_\infty)$ the last equation becomes

$$Nu = -\frac{\partial \theta(t,0)}{\partial \zeta}.$$

Since the significant velocity and temperature variations in the fluid are confined to the region adjacent to the disk, we define the thickness of these layers by certain standard measures [7]. For the tangential direction, we define a displacement thickness as

$$\delta_{dis} = \int_0^\infty G\, d\zeta.$$

As a measure of the extent of the thermal layer, we may introduce a thermal thickness based on the temperature excess above the ambient, then,

$$\delta_t = \int_0^\infty \theta\, d\zeta.$$

The action of viscosity in the fluid adjacent to the disk tends to set up a tangential shear stress $\overline{\tau}_\phi$ which opposes the rotation of the disk. There is also a surface shear stress $\overline{\tau}_r$ in the radial direction which, practically speaking, is of lesser importance than is the tangential stress. In terms of the variables of the analysis and by applying Newtonian shear formula [7], the expressions of $\overline{\tau}_\phi$ and $\overline{\tau}_r$ are respectively given as

$$\frac{\overline{\tau}_\phi}{\rho r(v\omega^3)^{1/2}} = \tau_\phi = \frac{\partial G(t,0)}{\partial \zeta},$$
$$\frac{\overline{\tau}_r}{\rho r(v\omega^3)^{1/2}} = \tau_r = \frac{\partial F(t,0)}{\partial \zeta}.$$

3. The numerical solution

Numerical solution for the governing non-linear equations (6)–(9) with the conditions given by Eq. (10), using the finite-difference approximations, leads to a numerical oscillation problem resulting from the discontinuity between the initial and boundary conditions (10a) and (10b). The same discontinuity occurs between the initial and boundary conditions for the energy problem (see Eq. (13)). A solution for this numerical problem is achieved by using proper coordinate transformations, as suggested by Ames [13] for similar problems. Expressing Eqs. (6)–(9) and (12) in terms of the modified coordinate $\eta = \zeta/2\sqrt{(t)}$ we get

$$\frac{\partial H}{\partial \eta} + 4\sqrt{(t)}F = 0, \tag{14}$$

$$\frac{\partial F}{\partial t} - \frac{\eta}{2t}\frac{\partial F}{\partial \eta} - \frac{1}{4t}\frac{\partial^2 F}{\partial \eta^2} + \frac{1}{2\sqrt{(t)}}H\frac{\partial F}{\partial \eta} + F^2 - G^2 + \frac{K}{8t}\left(\left(\frac{\partial F}{\partial \eta}\right)^2 + 3\left(\frac{\partial G}{\partial \eta}\right)^2 + 2F\frac{\partial^2 F}{\partial \eta^2}\right) = 0, \tag{15}$$

$$\frac{\partial G}{\partial t} - \frac{\eta}{2t}\frac{\partial G}{\partial \eta} - \frac{1}{4t}\frac{\partial^2 G}{\partial \eta^2} + \frac{1}{2\sqrt{(t)}}H\frac{\partial G}{\partial \eta} + 2FG - \frac{K}{4t}\left(\frac{\partial F}{\partial \eta}\frac{\partial G}{\partial \eta} - F\frac{\partial^2 G}{\partial \eta^2}\right) = 0, \tag{16}$$

$$\frac{\partial H}{\partial t} - \frac{\eta}{2t}\frac{\partial H}{\partial \eta} - \frac{1}{4t}\frac{\partial^2 H}{\partial \eta^2} + \frac{1}{2\sqrt{(t)}}H\frac{\partial H}{\partial \eta} + \frac{7}{16}\frac{K}{t\sqrt{t}}\frac{\partial H}{\partial \eta}\frac{\partial^2 H}{\partial \eta^2} - \frac{1}{2\sqrt{t}}\frac{\mathrm{d}P}{\mathrm{d}\eta} = 0, \tag{17}$$

$$\frac{\partial \theta}{\partial t} - \frac{\eta}{2t}\frac{\partial \theta}{\partial \eta} - \frac{1}{4t}\frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2\sqrt{(t)}}H\frac{\partial \theta}{\partial \eta} = 0. \tag{18}$$

The system of non-linear Partial differential equations (14)–(18) can be solved to determine the velocity, pressure, and temperature distributions. Here, Eqs. (14)–(16) and (18) are solved to determine the velocity and temperature distributions under the conditions given by Eqs. (10) and (13) while Eq. (17) may be solved to determine the pressure distribution if required. The Crank–Nicolson implicit method with a marching technique is applied [13]. The resulting system of difference equations has to be solved in the infinite domain $0 < \zeta < \infty$ and $0 < t < \infty$. A finite domain in the ζ-direction can be used instead with ζ chosen large enough to ensure that the solutions are not affected by imposing the asymptotic conditions at a finite distance. However, due to the suggested coordinate transformation, this finite domain diminishes with the progression of time and affects the accuracy of the numerical solution. To substitute for this problem, the modified Eqs. (15)–(17) are integrated from $t = 0$ to $t = 1$. Then, the solution obtained at $t = 1$ is used as the initial condition for integrating Eqs. (6)–(8) and (12) from $t = 1$ towards the steady state. It should be noted that the steady state solutions reported by Mithal [4] are reproduced by extending the transient solutions obtained here to the steady state. Also, the present results reduce to those reported by Attia [12] in the case of Newtonian fluid.

4. Results and discussion

The time growth of the axial velocity at infinity H_∞ for various values of the parameter K is shown in Fig. 1a. Increasing K decreases the axial flow towards the disk. Also, the parameter K has an interesting effect in reversing the direction of the axial velocity for some time which results in a crossover for the charts of H_∞ with time. The time at which the crossover point occurs increases as K increases. Fig. 1b presents the time variation of the azimuthal boundary layer thickness δ_{dis} for various values of K. It is clear from the figure that as K increases δ_{dis} increases with the presence of overshootings during the progress of time. The overshoots increase as K increases. Fig. 1c and d present the time variation of the tangential and radial shear stresses τ_ϕ and τ_r respectively for various values of K. Increasing K decreases the magnitude of τ_ϕ for small and moderate values of K, while increases its growth time as shown in Fig. 1c. Fig. 1d indicates an interesting effect for K in reversing the direction of τ_r during time. For small and moderate values of K, increasing K decreases τ_r, however, increases its growth time. For large values of K ($K = 2$), as shown in Fig. 1c and d, overshoots appear in both τ_ϕ and τ_r which indicates that the increase in K is expected to destabilize the laminar flow.

Fig. 1e presents the time development of the Nusselt number Nu for various values of the parameter K and for $Pr = 10$ which is appropriate for common liquids (water, oil, and so on) [14]. It is shown that as K increases Nu decreases. This is due to the fact that increasing K resists the axial flow towards the disk and then prevents the fluid at near-ambient temperature to be brought to the neighborhood of the surface of the disk which reduces the heat transfer (and hence, the

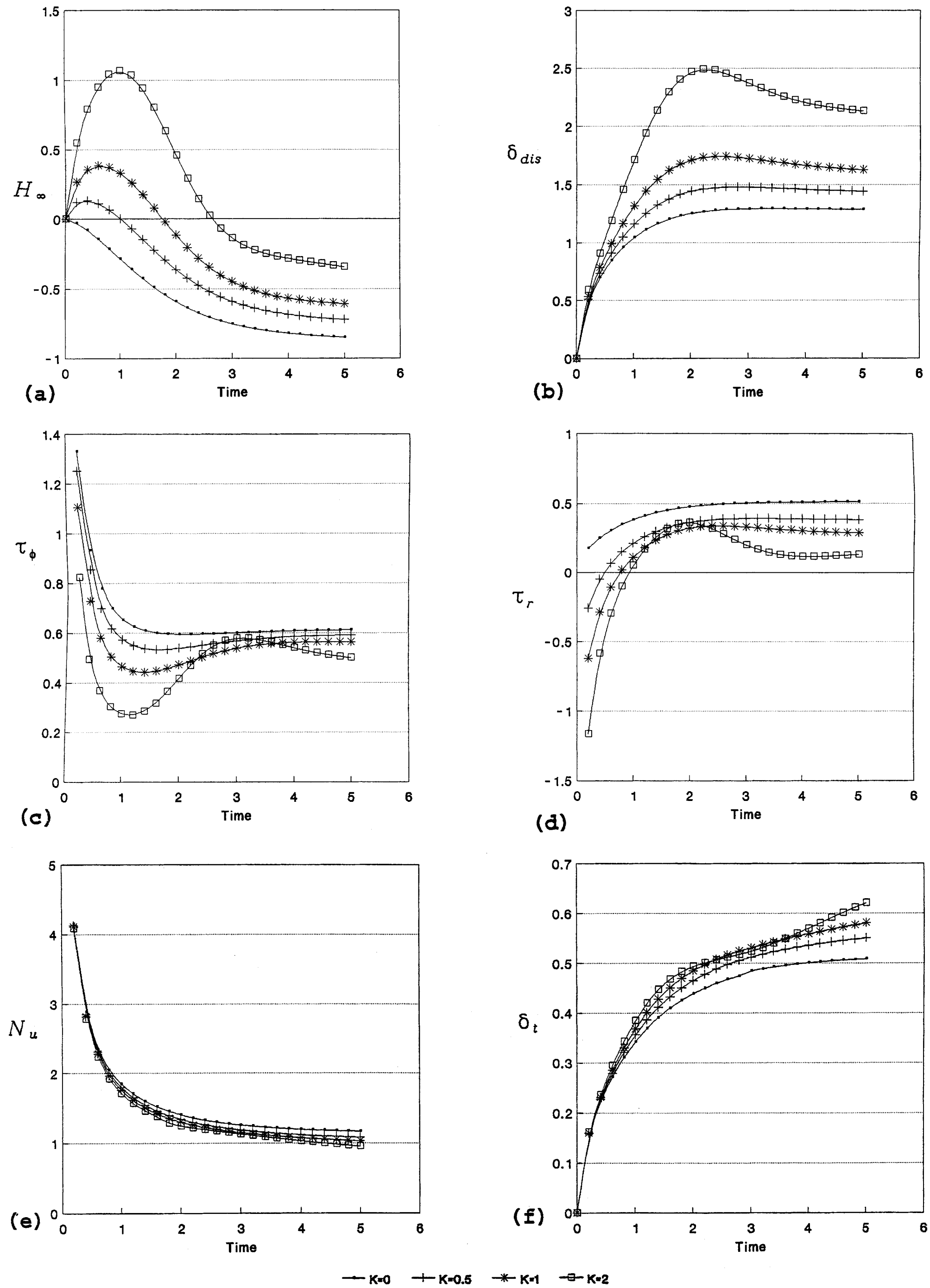

Fig. 1. Effect of K on the time variation of: (a) H_∞, (b) δ_{dis}, (c) τ_ϕ, (d) τ_r, (e) Nu, and (f) δ_t.

Nusselt number). Fig. 1f indicates the effect of increasing δ_t as a result of increasing K due to the influence of K in damping the axial flow towards the disk. It is also shown that, for large K ($K = 2$), a reduction in δ_t happens at a certain time which can be attributed to the reversal of the direction of H occurs with the progress in time (see Fig. 1a).

Fig. 2a–d present the steady state velocity components and temperature, F, G, H and θ respectively for various values of K and for $Pr = 10$. Fig. 2a indicates

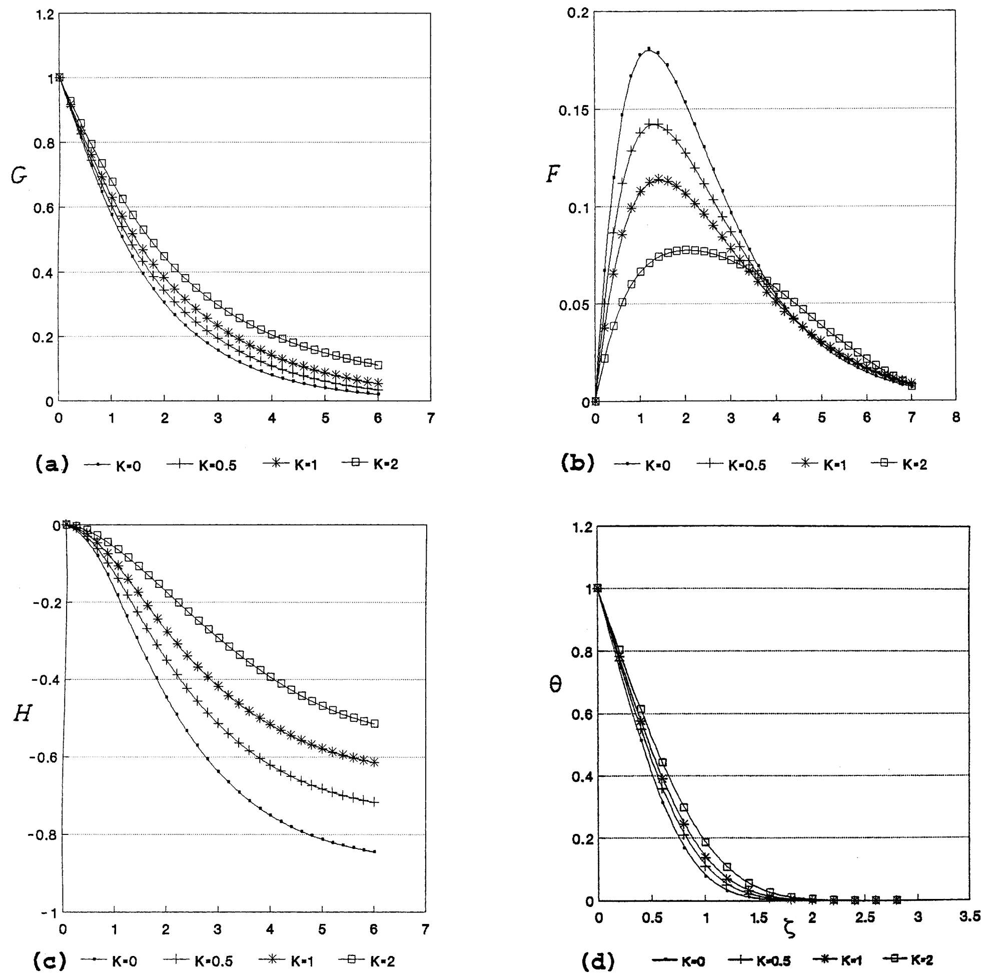

Fig. 2. Effect of K on the steady state profile of: (a) G, (b) F, (c) H, and (d) θ.

that increasing the parameter K decreases F for small and moderate values of ζ ($\zeta = 0$–4). However, for large values of ζ a crossover point that depends on K appears and an increment in K increases F. It is clear from Fig. 2b that increasing K increases G for all ζ. Fig. 2c shows that increasing the parameter K increases the resistance for the incoming axial flow and consequently reduces the axial velocity towards the disk H for all ζ due to the influence of K on reducing F. Fig. 2d indicates that increasing K increases the temperature θ for all ζ due to the effect of K in damping H.

5. Conclusions

In this paper the unsteady flow of a non-Newtonian fluid due to the uniform rotation of an infinite disk is studied with heat transfer. The effect of the non-Newtonian fluid characteristics, in terms of a parameter K, on the velocity and temperature distributions is considered. The parameter K has an interesting effect on reversing the direction of the axial velocity as well as the radial wall shear during time. Another important effect of K is the crossover appears in the H_∞–t and F–ζ charts. It can be concluded also that large values of the parameter K are expected to destabilize the laminar flow. Also, the parameter K has a reasonable effect on the heat transfer and temperature variations during time.

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